

On hidden symmetries of extremal Kerr-NUT-AdS-dS black holes

Jørgen Rasmussen

*Department of Mathematics and Statistics, University of Melbourne
Parkville, Victoria 3010, Australia*

`j.rasmussen@ms.unimelb.edu.au`

Abstract

It is well known that the Kerr-NUT-AdS-dS black hole admits two linearly independent Killing vectors and possesses a hidden symmetry generated by a rank-2 Killing tensor. The near-horizon geometry of an extremal Kerr-NUT-AdS-dS black hole admits four linearly independent Killing vectors, and we show how the hidden symmetry of the black hole itself is carried over by means of a modified Killing-Yano potential which is given explicitly. We demonstrate that the corresponding Killing tensor of the near-horizon geometry is reducible as it can be expressed in terms of the Casimir operators formed by the four Killing vectors.

1 Introduction

Isometries of spacetimes are generated by Killing vector fields, and one can associate a conserved quantity to each of these. A spacetime may also possess so-called hidden symmetries generated by higher-rank tensor fields. Following Carter's celebrated work [1] on the Kerr black hole, it was realized [2] that his new integral of motion is quadratic in momenta and generated by a rank-2 Killing tensor of the Kerr spacetime. Similar results have now been obtained for many classes of black holes and in various dimensions. We refer to the recent reviews [3, 4] for details and references.

Here we consider the family of extremal Kerr-NUT-AdS-dS black holes and our main focus is on the near-horizon geometries. Such geometries have attracted a lot of attention lately due to their role in the recently proposed Kerr/CFT correspondence [5]. Extensive lists of references on this topic can be found in [6].

First, we review the Kerr-NUT-AdS-dS black hole in the metric form discussed in [7]. This spacetime admits two linearly independent Killing vectors and possesses a hidden symmetry generated by a rank-2 Killing tensor. It is also recalled [8] that this Killing tensor follows from a Killing-Yano potential. We then determine the metric of the near-horizon geometry of the extremal Kerr-NUT-AdS-dS black hole and verify that it satisfies Einstein's field equations with the same cosmological constant as for the black hole itself. The isometry group, on the other hand, is enhanced as the near-horizon geometry admits four linearly independent Killing vectors. In the limiting procedure used to obtain the near-horizon geometry, the Killing-Yano potential diverges. However, a gauge freedom in the definition of the Killing-Yano potential allows us to construct a well-defined potential for the near-horizon geometry. We finally demonstrate that the corresponding rank-2 Killing tensor is reducible as it can be expressed in terms of the Casimir operators formed by the four Killing vectors of the near-horizon geometry.

2 Kerr-NUT-AdS-dS black holes

2.1 Geometry

Adopting the unit convention where $G = c = 1$, a general Kerr-NUT-AdS-dS black hole of mass M and with rotation parameter a is described by the metric [7]

$$d\hat{s}^2 = -\frac{\Delta_r}{\hat{r}^2 + y^2}(d\hat{t} + y^2 d\psi)^2 + \frac{\Delta_y}{\hat{r}^2 + y^2}(d\hat{t} - \hat{r}^2 d\psi)^2 + \frac{\hat{r}^2 + y^2}{\Delta_r} d\hat{r}^2 + \frac{\hat{r}^2 + y^2}{\Delta_y} dy^2 \quad (2.1)$$

where

$$\Delta_r = (\hat{r}^2 + a^2)\left(1 + \frac{\hat{r}^2}{\ell^2}\right) - 2M\hat{r}, \quad \Delta_y = (a^2 - y^2)\left(1 - \frac{y^2}{\ell^2}\right) + 2Ly \quad (2.2)$$

Here L is the NUT parameter (see also [9]) and the determinant \hat{g} of the metric only depends on \hat{r} and y as

$$\sqrt{-\hat{g}} = \hat{r}^2 + y^2 \quad (2.3)$$

This spacetime satisfies Einstein's field equations

$$\hat{G}_{\mu\nu} + \frac{3}{\ell^2} \hat{g}_{\mu\nu} = 0 \quad (2.4)$$

and is AdS (dS) for positive (negative) renormalized cosmological constant ℓ^{-2} . It reduces to Kerr for $L = \ell^{-2} = 0$. The horizons of the black hole are located at the positive zeros of Δ_r , and

the value of \hat{r} at the outer horizon is denoted by r_+ . The isometry group $U(1) \times U(1)$ of (2.1) is generated by the Killing vector fields

$$\{\partial_\psi\} \cup \{\partial_{\hat{t}}\} \quad (2.5)$$

2.2 Hidden symmetry

As observed in [8] (see also the reviews [4]), the metric (2.1) admits the so-called Killing-Yano potential (1-form)

$$\hat{\mathbf{b}} = \frac{y^2 - \hat{r}^2}{2} d\hat{t} - \frac{\hat{r}^2 y^2}{2} d\psi \quad (2.6)$$

This implies the existence of the principal conformal Killing-Yano tensor [10]

$$\hat{\mathbf{h}} = d\hat{\mathbf{b}} \quad (2.7)$$

and its Hodge dual, the Killing-Yano tensor [11]

$$\hat{\mathbf{f}} = *\hat{\mathbf{h}} = *d\hat{\mathbf{b}} \quad (2.8)$$

From this, one can construct a symmetric Killing tensor by contraction

$$\hat{K}_{\mu\nu} = \hat{f}_{\mu\lambda} \hat{f}_\nu{}^\lambda \quad (2.9)$$

This tensor $\hat{\mathbf{K}}$ is responsible for the hidden symmetry associated with the conserved quantity $\hat{K}^{\mu\nu} \hat{p}_\mu \hat{p}_\nu$ quadratic in the momenta \hat{p}_μ . Considering the construction of $\hat{\mathbf{K}}$ above, it appears natural to refer to the entire quadruplet $(\hat{\mathbf{b}}, \hat{\mathbf{h}}, \hat{\mathbf{f}}, \hat{\mathbf{K}})$ as the generator of the hidden symmetry.

It is straightforward to verify that

$$\hat{\mathbf{K}} = \hat{\mathbf{Q}} + \hat{r}^2 \hat{\mathbf{g}} \quad (2.10)$$

where $\hat{\mathbf{g}}$ is the metric tensor while the components of $\hat{\mathbf{Q}}$ in the ordered basis $\{\hat{t}, \hat{r}, \psi, y\}$ are given by

$$[\hat{Q}_{\mu\nu}] = \begin{pmatrix} \Delta_r & 0 & \Delta_r y^2 & 0 \\ 0 & -\frac{(\hat{r}^2 + y^2)^2}{\Delta_r} & 0 & 0 \\ \Delta_r y^2 & 0 & \Delta_r y^4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.11)$$

With raised indices, the components are

$$[\hat{Q}^{\mu\nu}] = \frac{1}{\Delta_r} \begin{pmatrix} \hat{r}^4 & 0 & \hat{r}^2 & 0 \\ 0 & -\Delta_r^2 & 0 & 0 \\ \hat{r}^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.12)$$

3 Extremal Kerr-NUT-AdS-dS black holes

When the inner and outer horizons of the black hole (2.1) coalesce, the black hole is said to be extremal. This happens when the otherwise single pole r_+ of Δ_r is a double pole in which case

$$\Delta_r(r_+) = \Delta'_r(r_+) = 0 \quad (3.1)$$

We denote the value of \hat{r} at this single horizon by \bar{r} . The conditions (3.1) can be used to express M and a^2 at extremality in terms of \bar{r}

$$M = \frac{\bar{r}(1 + \frac{\bar{r}^2}{\ell^2})^2}{1 - \frac{\bar{r}^2}{\ell^2}}, \quad a^2 = \frac{\bar{r}^2(1 + \frac{3\bar{r}^2}{\ell^2})}{1 - \frac{\bar{r}^2}{\ell^2}} \quad (3.2)$$

3.1 Near-horizon geometry

To describe the near-horizon geometry of an extremal Kerr-NUT-AdS-dS black hole, we introduce (in the spirit of [12]) the new coordinates t, r, ϕ

$$t = \frac{(\bar{r}^2 + a^2)\epsilon\hat{t}}{\bar{r}^2\bar{r}_0}, \quad r = \frac{\hat{r} - \bar{r}}{\epsilon\bar{r}_0}, \quad \phi = \bar{r}^2\psi - \hat{t} \quad (3.3)$$

where

$$\bar{r}_0^2 = \frac{(\bar{r}^2 + a^2)(1 - \frac{\bar{r}^2}{\ell^2})}{1 + \frac{6\bar{r}^2}{\ell^2} - \frac{3\bar{r}^4}{\ell^4}} \quad (3.4)$$

The transformation of the radial coordinate facilitates zooming in on the near-horizon region while the accompanying transformations ensure that the line element (2.1) is well-defined in the limit $\epsilon \rightarrow 0$. The near-horizon geometry is then obtained by taking the limit $\epsilon \rightarrow 0$ in which case the metric becomes¹

$$d\bar{s}^2 = \frac{\bar{r}_0^2(\bar{r}^2 + y^2)}{\bar{r}^2 + a^2}(-r^2 dt^2 + \frac{1}{r^2} dr^2) + \frac{\Delta_y}{\bar{r}^2 + y^2}(d\phi + \bar{k}r dt)^2 + \frac{\bar{r}^2 + y^2}{\Delta_y} dy^2 \quad (3.5)$$

where

$$\bar{k} = \frac{2\bar{r}\bar{r}_0^2}{\bar{r}^2 + a^2} \quad (3.6)$$

It is straightforward, albeit tedious, to verify explicitly that this metric satisfies Einstein's field equations

$$G_{\mu\nu} + \frac{3}{\ell^2}g_{\mu\nu} = 0 \quad (3.7)$$

We find that the determinant g of the metric (3.5) only depends on the coordinate y as

$$\sqrt{-g} = \frac{\bar{r}_0^2(\bar{r}^2 + y^2)}{\bar{r}^2 + a^2} \quad (3.8)$$

and it is noted that the metric describes the near-horizon geometry of the Kerr black hole when $N = \ell^{-2} = 0$.

The exact isometry group of the spacetime (3.5) is generated by the Killing vectors

$$\{\mathbf{P} = \partial_\phi\} \cup \{\mathbf{T} = \partial_t, \mathbf{D} = t\partial_t - r\partial_r, \mathbf{F} = (t^2 + \frac{1}{r^2})\partial_t - 2tr\partial_r - \frac{2\bar{k}}{r}\partial_\phi\} \quad (3.9)$$

It follows that the isometry group $U(1) \times U(1)$ of the Kerr-NUT-AdS-dS black hole is enhanced to $U(1) \times SL(2)$ in the near-horizon geometry.

¹Using different coordinates, a similar limit and near-horizon geometry is discussed in [13], although Einstein's field equations (3.7) are not addressed there.

3.2 Hidden symmetry

After the substitution (3.3), the Killing-Yano potential $\hat{\mathbf{b}}$ (2.6) *diverges* in the limit $\epsilon \rightarrow 0$. This could indicate that the near-horizon geometry does not possess a hidden symmetry admitting a Killing-Yano potential, at least not one carried over from the original Kerr-NUT-AdS-dS spacetime. This is not the case, though, as the divergence problem can be resolved and a well-defined Killing-Yano potential can be constructed. To demonstrate this, we use that a Killing-Yano potential is defined only up to a *constant* 1-form. This corresponds to gauge invariance of the corresponding Killing-Yano and Killing tensors. We thus find that

$$\hat{\mathbf{b}} + \frac{1}{2}(\alpha d\hat{t} + \beta d\psi) \quad (3.10)$$

with α and β constant, remains well-defined in the limit $\epsilon \rightarrow 0$ provided

$$-\bar{r}^4 + \alpha\bar{r}^2 + \beta = 0 \quad (3.11)$$

The 1-form thereby obtained is given by

$$\mathbf{b} = -\frac{\bar{k}(\bar{r}^2 + y^2)r}{2}dt - \frac{y^2}{2}d\phi \quad (3.12)$$

and we have verified explicitly that the tensor \mathbf{K} constructed as in (2.7)-(2.9) by

$$\mathbf{h} = d\mathbf{b}, \quad \mathbf{f} = *\mathbf{h} = *d\mathbf{b}, \quad K_{\mu\nu} = f_{\mu\lambda}f_{\nu}{}^{\lambda} \quad (3.13)$$

is indeed a Killing tensor. The 1-form \mathbf{b} is therefore the sought-after Killing-Yano potential for the Killing tensor \mathbf{K} . We conclude that the quadruplet $(\mathbf{b}, \mathbf{h}, \mathbf{f}, \mathbf{K})$ generates a hidden symmetry of the near-horizon geometry (3.5) of a Kerr-NUT-AdS-dS black hole.

It is observed that the Killing tensor \mathbf{K} decomposes as

$$\mathbf{K} = \mathbf{Q} + \bar{r}^2 \mathbf{g} \quad (3.14)$$

where \mathbf{g} is the metric tensor of the near-horizon geometry (3.5) while the components of \mathbf{Q} are given by

$$[Q_{\mu\nu}] = -\sqrt{-g} \begin{pmatrix} -r^2 & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.15)$$

Aside from the y -dependent factor $\sqrt{-g}$, this is recognized as the metric tensor of AdS_2 in the coordinates t, r . With raised indices, the components of \mathbf{Q} only depend on the radial coordinate and are given by

$$[Q^{\mu\nu}] = \begin{pmatrix} \frac{\bar{r}^2 + a^2}{\bar{r}_0^2 r^2} & 0 & -\frac{2\bar{r}}{r} & 0 \\ 0 & -\frac{(\bar{r}^2 + a^2)r^2}{\bar{r}_0^2} & 0 & 0 \\ -\frac{2\bar{r}}{r} & 0 & \frac{4\bar{r}^2 \bar{r}_0^2}{\bar{r}^2 + a^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.16)$$

Since the metric tensor is multiplied by a *constant* in the decomposition (3.14), the ‘essential’ part of the hidden symmetry is generated by \mathbf{Q} implying that the metric-tensor term can be ignored when analyzing the hidden symmetry generated by \mathbf{K} .

We find that this hidden symmetry is *reducible* in the sense that $Q^{\mu\nu}$ can be expressed in terms of the Killing vectors (3.9) by

$$Q^{\mu\nu} = \gamma(\mathbf{T} \otimes \mathbf{F} + \mathbf{F} \otimes \mathbf{T} - 2\mathbf{D} \otimes \mathbf{D})^{\mu\nu} - \frac{2}{\gamma\bar{r}^2}(\mathbf{P} \otimes \mathbf{P})^{\mu\nu} \quad (3.17)$$

where

$$\gamma = \frac{\bar{r}^2(1 + \bar{r}^2/\ell^2)}{\bar{r}_0^2(1 - \bar{r}^2/\ell^2)} \quad (3.18)$$

The Killing tensor \mathbf{Q} is thus a linear combination of the quadratic Casimir operators formed out of the two sets of Killing vectors (3.9). The various constants appearing in this decomposition can be absorbed by rescaling the Killing vectors.

Note added

During the completion of this work, the paper [14] appeared. It has some overlap with the present work as it also discusses the hidden symmetry of the near-horizon geometry of the extremal Kerr black hole. The reducibility of the corresponding Killing tensor is discussed using Poisson brackets in the context of particle dynamics near the extremal Kerr throat. The paper [14] does not, however, consider the near-horizon geometry of the general Kerr-NUT-AdS-dS black hole, and it does not address the Killing-Yano potential underlying the hidden symmetry. On the other hand, it proposes an $\mathcal{N} = 2$ supersymmetric extension of the particle dynamics and it may be of interest to generalize this to the Kerr-NUT-AdS-dS scenario considered here.

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